

REPRESENTATION OF SELF-SIMILAR GAUSSIAN PROCESSES

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ABSTRACT. We develop the canonical Volterra representation for a self-similar Gaussian process by using the Lamperti transformation of the corresponding stationary Gaussian process, where this latter one admits a canonical integral representation under the assumption of pure non-determinism. We apply the representation obtained to the equivalence in law for self-similar Gaussian processes.

Mathematics Subject Classification (2010): 60G15, 60G18, 60G22.

Keywords: Self-similar processes; Gaussian processes; canonical Volterra representation; Lamperti transformation; stationary Gaussian process; equivalence in law; homogeneous kernels.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we will construct the canonical Volterra representation for a given self-similar centered Gaussian processes. The role of the canonical Volterra representation which was first introduced by Levy in [15] and [16], and later developed by Hida in [9], is to provide an integral representation for a Gaussian process X in terms of a Brownian motion W and a non-random Volterra kernel k such that the expression

$$X_t = \int_0^t k(t, s) dW_s$$

holds for all t and the Gaussian processes X and W generate the same filtration. It is known, see [3] and [15], that if the kernel k satisfies the homogeneity property for some degree α , i.e. $k(at, as) = a^\alpha k(t, s)$, $a > 0$, the Gaussian process X is self-similar with index $\alpha + \frac{1}{2}$. Thus, the main goal of this paper is to give, under some suitable conditions, a general construction of the canonical Volterra representation for self-similar Gaussian processes, and which also guaranties the homogeneity property of the kernel. In section 2, the linear Lamperti transform that defines the one-one correspondence between stationary processes and self-similar processes, will be used to express the explicit form of the canonical Volterra representation for self-similar Gaussian processes in the light of the classical canonical representation of the stationary processes given by Karhunen in [12]. In section 3, we give an application of the representation obtained to a Gaussian process equivalent in law to the self-similar Gaussian process.

In our mathematical settings, we take $T > 1$ to be a fixed time horizon, and on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a centered Gaussian process

Date: July 24, 2014.

The author wishes to thank Tommi Sottinen for discussions and helpful comments. The author also thanks the Finnish Doctoral Programme in Stochastics and Statistics and the Finnish Cultural Foundation for financial support.

$X = (X_t; t \in [0, T])$ that enjoys the self-similarity property for some $\beta > 0$, i.e.

$$(X_{at})_{0 \leq t \leq T/a} \stackrel{d}{=} (a^\beta X_t)_{0 \leq t \leq T}, \quad \text{for all } a > 0,$$

where $\stackrel{d}{=}$ denotes equality in distributions, or equivalently ,

$$r(t, s) = \mathbb{E}(X_t X_s) = T^{2\beta} r\left(\frac{t}{T}, \frac{s}{T}\right), \quad 0 \leq t, s \leq T. \quad (1.1)$$

In particular, we have $r(t, t) = t^{2\beta} \mathbb{E}(X_1^2)$, which is finite and continuous function at every (t, t) in $[0, T]^2$, and therefore, is continuous at every $(t, s) \in [0, T]^2$, see [17]. A consequence of the continuity of the covariance function r is that X is mean-continuous.

We denote by $H_X(t)$ the closed linear subspace of $L^2([0, T])$ generated by Gaussian random variables X_s for $s \leq t$, and by $(\mathcal{F}_t^X)_{t \in [0, T]}$, where $\mathcal{F}_t^X := \sigma(X_s, s \leq t)$, the completed natural filtration of X . We call the *Volterra representation* of X the integral representation of the form

$$X_t = \int_0^t k(t, s) dW_s, \quad t \in [0, T], \quad (1.2)$$

where $W = (W_t; t \in [0, T])$ is a standard Brownian motion and the kernel $k(t, s)$ is a Volterra kernel, i.e. a measurable function on $[0, T] \times [0, T]$ that satisfies $\int_0^T \int_0^t k(t, s)^2 ds dt < \infty$, and $k(t, s) = 0$ for $s > t$. The Gaussian process X with such representation is called a *Gaussian Volterra process*, provided with k and W .

Moreover, the Volterra representation is said to be *canonical* if the *canonical property*

$$\mathcal{F}_t^X = \mathcal{F}_t^W$$

holds for all t , or equivalently

$$H_X(t) = H_W(t), \quad \text{for all } t. \quad (1.3)$$

Remark 1.1. (i) An equivalent to the canonical property is that if there exists a random variable $\eta = \int_0^T \phi(s) dW_s$, $\phi \in L^2([0, T])$, such that it is independent of X_t for all $0 \leq t \leq T$, i.e. $\int_0^t k(t, s) \phi(s) ds = 0$, one has $\phi \equiv 0$. This means that the family $\{k(t, \cdot), 0 \leq t \leq T\}$ is free and spans a vector space that is dense in $L^2([0, T])$. If we associate with the canonical kernel k a Volterra integral operator \mathcal{K} defined on $L^2([0, T])$ by $\mathcal{K}\phi(t) = \int_0^t k(t, s) \phi(s) ds$, it follows from the canonical property (1.3) that \mathcal{K} is injective and $\mathcal{K}(L^2([0, T]))$ is dense in $L^2([0, T])$. The covariance integral operator \mathcal{R} associated with the kernel $r(t, s)$ has the decomposition $\mathcal{R} = \mathcal{K}\mathcal{K}^*$, where \mathcal{K}^* is the adjoint operator of \mathcal{K} . In this case, the covariance r is factorable, i.e.

$$r(t, s) = \int_0^{t \wedge s} k(t, u) k(s, u) du.$$

(ii) A special property for a Volterra integral operator is that it has no eigenvalues, see [7].

2. THE CANONICAL VOLTERRA REPRESENTATION AND SELF-SIMILARITY

The Gaussian process X is β -self-similar, and according to Lamperti [14], it can be transformed into a stationary Gaussian process Y defined by:

$$Y(t) := e^{-\beta t} X(e^t), \quad t \in (-\infty, \log T]. \quad (2.1)$$

Conversely, X can be recovered from Y by the inverse Lamperti transformation

$$X(t) = t^\beta Y(\log t), \quad t \in [0, T]. \quad (2.2)$$

It is obvious that the mean-continuity of the process Y follows from the fact that

$$\mathbb{E}(Y_t - Y_s)^2 = 2 \left(r(1, 1) - e^{-(t-s)\beta} r(e^{t-s}, 1) \right)$$

converges to zero when t approaches s . As was shown by Hida & Hitsuda (§3, [10]), which is a well-known classical result that has been first established by Karhunen (§3, Satz 5, [12]), the stationary Gaussian process Y admits the canonical representation

$$Y_t = \int_{-\infty}^t G_T(t-s) dW_s^*, \quad (2.3)$$

where G_T is a measurable function that belongs to $L^2(\mathbb{R}, du)$ such that $G_T(u) = 0$ when $u < 0$, and W^* is a standard Brownian motion satisfying the canonical property, i.e., $H_Y(t) = H_{W^*}(t)$, $t \in [0, T]$. A necessary and sufficient condition for the existence of the representation (2.3) is that Y is purely non-deterministic. Following Cramer [4], a process Z is purely non-deterministic if and only if the condition

$$\bigcap_t H_Z(t) = \{0\}, \quad (C)$$

is fulfilled, where $\{0\}$ is the L^2 -subspace spanned by the constants. The condition (C) means that the remote past is trivial, i.e. \mathcal{F}_{0+}^Z is trivial; see also [8], [10] and [12].

Next, we shall extend the property of pure non-determinism to the self-similar centered Gaussian process X .

Theorem 2.1. *The self-similar centered Gaussian process $X = (X_t; t \in [0, T])$ satisfies the condition (C) if and only if there exist a standard Brownian motion W and a Volterra kernel k such that X has the representation*

$$X_t = \int_0^t k(t, s) dW_s, \quad (2.4)$$

where the Volterra kernel k is defined by

$$k(t, s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right), \quad s < t, \quad (2.5)$$

for some function $F \in L^2(\mathbb{R}_+, du)$ independent of β , with $F(u) = 0$ for $1 < u$.

Moreover, $H_X(t) = H_W(t)$ holds for each t .

Remark 2.2. In the case where the process X is *trivial self-similar*, i.e. $X_t = t^\beta W_1$, $0 \leq t \leq T$, the condition (C) is not satisfied since $\bigcap_{t \in (0, T)} H_X(t) = H_W(1)$. Thus, X has no Volterra representation in this case.

Proof. The fact that X is purely non-deterministic is equivalent to that Y is purely non-deterministic since

$$\bigcap_{t \in (0, T)} H_X(t) = \bigcap_{t \in (0, T)} H_Y(\log t) = \bigcap_{t \in (-\infty, \log T)} H_Y(t).$$

Thus Y admits the representation (2.3) for some square integrable kernel G_T and a standard Brownian motion W^* . By the inverse Lamperti transformation, we obtain

$$X(t) = \int_{-\infty}^{\log t} t^\beta G_T(\log t - s) dW_s^* = \int_0^t t^\beta s^{-\frac{1}{2}} G_T\left(\log \frac{t}{s}\right) dW_s,$$

where $dW_s = s^{\frac{1}{2}} dW_{\log s}^*$. We take the Volterra kernel k to be defined as $k(t, s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right)$, where $F(u) = u^{-\frac{1}{2}} G_T(\log u^{-1}) \in L^2(\mathbb{R}_+, du)$ vanishing when $u < 1$ since $G_T(u) = 0$ when $u < 0$, i.e. for $t < s$, we have $F(\frac{s}{t}) = 0$, and then, $k(t, s) = 0$. Indeed,

$$\int_0^\infty F(u)^2 du = \int_0^\infty G_T(\log u^{-1})^2 \frac{du}{u} = \int_{-\infty}^\infty G_T(v)^2 dv < \infty,$$

and

$$\begin{aligned} \int_0^T \int_0^t F\left(\frac{s}{t}\right)^2 ds dt &= \int_0^T t dt \int_0^1 F(u)^2 du \\ &= \int_0^T t dt \int_0^\infty G_T(v)^2 dv < \infty. \end{aligned}$$

Thus,

$$\int_0^T \int_0^t t^{2\beta-1} F\left(\frac{s}{t}\right)^2 ds dt = \left(\int_0^T t^{2\beta} dt \right) \left(\int_0^1 F(u)^2 du \right) < \infty$$

Considering the closed linear subspace $H_{dW}(t)$ of $L^2([0, T])$ that is generated by $W_s - W_u$ for all $u \leq s \leq t$, we have $H_{dW}(t) = H_W(t)$ since $W_0 = 0$, and therefore, the canonical property follows from the equalities

$$H_X(t) = H_Y(\log t) = H_{dW^*}(\log t) = H_{dW}(t) = H_W(t).$$

□

Example 2.3 (Fractional Brownian motion). The fractional Brownian motion (fBm) on $[0, T]$ with index $H \in (0, 1)$ is a centered Gaussian process $B^H = (B_t; 0 \leq t \leq T)$ with the covariance function $R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$. The fBm is H -self-similar, and following [1] and [5], it admits the canonical Volterra representation with the canonical kernel

$$\begin{aligned} k_H(t, s) &= c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad \text{for } H > \frac{1}{2}, \\ k_H(t, s) &= d_H \left(\left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \right. \\ &\quad \left. - \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right), \quad \text{for } H < \frac{1}{2}, \end{aligned}$$

where $c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$, $d_H = \left(\frac{2H}{(1-2H)B(1-2H, H+\frac{1}{2})} \right)^{\frac{1}{2}}$, here B denotes the Beta function. So, the function F that corresponds to the canonical Volterra representation of fBm has the expressions:

$$F(u) = c_H \left(u^{\frac{1}{2}-H} \int_u^1 (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz \right), \quad \text{for } H > \frac{1}{2},$$

and

$$F(u) = d_H \left(\left(\frac{1}{u} - 1 \right)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) (u)^{\frac{1}{2}-H} \int_u^1 z^{H-\frac{3}{2}} (z-u)^{H-\frac{1}{2}} dz \right),$$

for $H < \frac{1}{2}$.

A function $f(t, s)$ is said to be homogeneous with degree α if the equality

$$f(at, as) = a^\alpha f(t, s), \quad a > 0,$$

holds for all t, s in $[0, T]$. From the expression (2.5) of the canonical kernel, it is easy to see that k is homogeneous with degree $\beta - \frac{1}{2}$, i.e. $k(t, s) = T^{\beta-\frac{1}{2}} k(\frac{t}{T}, \frac{s}{T})$, for all $s < t \in [0, T]$.

Given X with the canonical Volterra representation (2.4), let \mathcal{U} to be a bounded unitary endomorphism on $L^2([0, T])$ with adjoint $\mathcal{U}^* = \mathcal{U}^{-1}$, and define the process $B = (B)_t := (\mathcal{U}^*(W))_t$ for each $t \in [0, T]$. Indeed, B is a standard Brownian motion since the Gaussian measure is preserved under the unitary transformations. With the notation $k_t(\cdot) := k(t, \cdot)$, the Gaussian process associated with the kernel $(\mathcal{U}k_t)(s)$ and the standard Brownian motion B has same law as X . For the covariance operator, we write

$$\mathcal{R} = \mathcal{K} \mathcal{K}^* = \mathcal{K} \mathcal{U}^* \mathcal{U} \mathcal{K}^* = (\mathcal{K} \mathcal{U}^*)(\mathcal{K} \mathcal{U}^*)^*,$$

where the operator $\mathcal{K} \mathcal{U}^*$ is defined by

$$(\mathcal{K} \mathcal{U}^*)\phi(t) = \int_0^t k(t, s) (\mathcal{U}^*\phi)(s) ds = \int_0^T (\mathcal{U}k_t)(s) \phi(s) ds, \quad \phi \in L^2([0, T]).$$

The associated Gaussian process has then the integral representation $\int_0^T (\mathcal{U}k_t)(s) dB_s$ for all $t \in [0, T]$.

Corollary 2.4. *For any bounded unitary endomorphism \mathcal{U} on $L^2([0, T])$, the homogeneity of k is preserved under \mathcal{U} .*

Proof. Let \mathcal{U} be a bounded unitary endomorphism on $L^2([0, T])$, and let the scaling operator $\mathcal{S}f(t) = T^{\frac{1}{2}}f(Tt)$ with adjoint $\mathcal{S}^*f(t) = T^{-\frac{1}{2}}f(\frac{t}{T})$ to be defined for all $f \in L^2([0, T])$. The homogeneity of k means that

$$k_t(s) = T^\beta (\mathcal{S}^*k_{\frac{t}{T}})(s),$$

then we have

$$\mathcal{U}k_t(s) = T^\beta (\mathcal{U} \mathcal{S}^*k_{\frac{t}{T}})(s) = T^{\beta-\frac{1}{2}} (\mathcal{S} \mathcal{U} \mathcal{S}^*k_{\frac{t}{T}})(\frac{s}{T}).$$

To show the equality $\mathcal{S}\mathcal{U}\mathcal{S}^*k_{\frac{t}{T}} = \mathcal{U}k_{\frac{t}{T}}$, we will use the Mellin transform

$$\begin{aligned}
\int_0^\infty (\mathcal{S}\mathcal{U}\mathcal{S}^*k_{\frac{t}{T}})(s) s^{p-1} ds &= \int_0^\infty (\mathcal{U}\mathcal{S}^*k_{\frac{t}{T}})(s) (\mathcal{S}^*s^{p-1}) ds \\
&= T^{\frac{1}{2}-p} \int_0^\infty (\mathcal{U}\mathcal{S}^*k_{\frac{t}{T}})(s) s^{p-1} ds \\
&= T^{\frac{1}{2}-p} \int_0^\infty (\mathcal{S}^*k_{\frac{t}{T}})(s) (\mathcal{U}^*s^{p-1}) ds \\
&= T^{-p} \int_0^\infty k_{\frac{t}{T}}\left(\frac{s}{T}\right) (\mathcal{U}^*s^{p-1}) ds \\
&= \int_0^\infty k_{\frac{t}{T}}(u) (\mathcal{U}^*u^{p-1}) du = \int_0^\infty \mathcal{U}k_{\frac{t}{T}}(u) u^{p-1} du,
\end{aligned}$$

and the uniqueness property of the Mellin transform implies that

$$\mathcal{S}\mathcal{U}\mathcal{S}^*k_{\frac{t}{T}} = \mathcal{U}k_{\frac{t}{T}}.$$

□

Remark 2.5. The fact that the β -self-similar Gaussian process X satisfies the condition (C), guaranties the existence of the canonical kernel k which is homogeneous with degree $\beta - \frac{1}{2}$, and its homogeneity is preserved under unitary transformation. If we consider again the example in Remark 2.2, one has the representation

$$X_t = \int_0^T t^\beta 1_{[0,1]}(s) dW_s, 0 \leq t \leq T,$$

where $1_{[0,1]}(s)$ is the indicator function. In this case, we see that the kernel $t^\beta 1_{[0,1]}(s)$ does not satisfy the homogeneity property of any degree.

3. APPLICATION TO THE EQUIVALENCE IN LAW

In this section, we shall emphasize the self-similarity property under the equivalence of laws of Gaussian processes. First, We recall the results shown by Hida-Hitsuda in the case of Brownian motion, see [10] and [11]. Following Hitsuda's representation theorem, a centered Gaussian process $\widetilde{W} = (\widetilde{W}_t; t \in [0, T])$ is equivalent in law to a standard Brownian motion $W = (W_t; t \in [0, T])$ if and only if \widetilde{W} can be represented in a unique way by

$$\widetilde{W}_t = W_t - \int_0^t \int_0^s l(s, u) dW_u ds, \quad (3.1)$$

where $l(s, u)$ is a Volterra kernel, i.e.

$$\int_0^T \int_0^t l(t, s)^2 ds dt < \infty, \quad l(t, s) = 0 \quad \text{for } t < s, \quad (3.2)$$

and such that the equality $H_{\widetilde{W}}(t) = H_W(t)$ holds for each t . We note here that the uniqueness of the canonical decomposition (3.1) is in the sense that if l' is a Volterra kernel and $W' = (W'_t; t \in [0, T])$ is a standard Brownian motion such that for $0 \leq t \leq T$

$$W'_t - \int_0^t \int_0^s l'(s, u) dW'_u ds = W_t - \int_0^t \int_0^s l(s, u) dW_u ds,$$

then $l = l'$ and $W = W'$.

If we denote by \mathbb{P} and $\widetilde{\mathbb{P}}$ the laws of W and \widetilde{W} respectively, these two processes are equivalent in law if \mathbb{P} and $\widetilde{\mathbb{P}}$ are equivalent, and the Radon-Nikodym density is given by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ \int_0^T \int_0^s l(s, u) dW_u dW_s - \frac{1}{2} \int_0^T \left(\int_0^s l(s, u) dW_s \right)^2 ds \right\}.$$

The centered Gaussian process \widetilde{W} is a standard Brownian motion under $\widetilde{\mathbb{P}}$ with $\widetilde{\mathbb{E}}(\widetilde{W}_t \widetilde{W}_s) = \mathbb{E}(W_t W_s)$, hence, it is self-similar with index $\frac{1}{2}$ under $\widetilde{\mathbb{P}}$. It follows from (3.1) that the covariance of \widetilde{W} under \mathbb{P} has the form of

$$\begin{aligned} \mathbb{E}(\widetilde{W}_t \widetilde{W}_s) &= t \wedge s - \int_0^{t \wedge s} \int_u^s l(v, u) dv du - \int_0^{t \wedge s} \int_u^t l(v, u) dv du \\ &\quad + \int_0^t \int_0^s \int_0^{v_1 \wedge v_2} l(v_1, u) l(v_2, u) du dv_1 dv_2. \end{aligned}$$

The Hitsuda representation can be extended to the class of the canonical Gaussian Volterra processes, see [2] and [20]. A centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0, T])$ is equivalent in law to a Gaussian Volterra process X if and only if there exists a unique centered Gaussian process, namely \widetilde{W} , satisfying (3.1) and (3.2), and such that

$$\widetilde{X}_t = \int_0^t k(t, s) d\widetilde{W}_s = X_t - \int_0^t k(t, s) \int_0^s l(s, u) dW_u ds, \quad (3.3)$$

where the kernel $k(t, s)$ and the standard Brownian motion stand for (1.2), the canonical Volterra representation of X . Moreover, we have $H_{\widetilde{X}}(t) = H_X(t)$ for all t .

Under the condition (C), the kernel k is $(\beta - \frac{1}{2})$ -homogeneous, and the centered Gaussian process \widetilde{X} is β -self-similar under $\widetilde{\mathbb{P}}$ since \widetilde{W} is a standard Brownian motion. It is obvious that if \widetilde{X} has same law as X , it is β -self-similar under \mathbb{P} , and this condition is also necessary, see [18]. However, in the next proposition, we will use the homogeneity property of the Volterra kernel l as a necessary and sufficient condition for the self-similarity for the process \widetilde{X} , and equivalently for \widetilde{W} , under the law \mathbb{P} .

Proposition 3.1. *Let $X = (X_t; t \in [0, T])$ be a centered β -self-similar Gaussian process satisfying the condition (C), then*

- (i) *a centered Gaussian process $\widetilde{X} = (\widetilde{X}_t; t \in [0, T])$ is equivalent in law to X if and only if \widetilde{X} admits a representation of the form of*

$$\widetilde{X}_t = X_t - t^{\beta - \frac{1}{2}} \int_0^t z(t, s) dW_s, \quad 0 \leq t \leq T, \quad (3.4)$$

where W is a standard Brownian motion on $[0, T]$, and the kernel $z(t, s)$ is independent of β and expressed by

$$z(t, s) = \int_s^t F\left(\frac{u}{t}\right) l(u, s) du, \quad s < t,$$

for a Volterra kernel l and some function $F \in L^2(\mathbb{R}_+, du)$ vanishing on $(1, \infty]$.

- (ii) *In addition, \widetilde{X} is β -self-similar if and only if $l \equiv 0$.*

For the proof, we need the following lemma.

Lemma 3.2. *If a Volterra kernel on $[0, T] \times [0, T]$ is homogeneous with degree (-1) , then it vanishes on $[0, T] \times [0, T]$.*

Proof. Let a Volterra kernel h be (-1) -homogeneous. Combining the square integrability and the homogeneity property $h(t, s) = \frac{1}{a} h(\frac{t}{a}, \frac{s}{a})$, $a > 0$, $0 \leq s < t \leq T$, yields

$$\int_0^T \int_0^t h(t, s)^2 ds dt = \int_0^{\frac{T}{a}} \int_0^{\frac{t}{a}} h\left(\frac{t}{a}, \frac{s}{a}\right)^2 \frac{1}{a^2} ds dt = \int_0^{\frac{T}{a}} \int_0^{t'} h(t', s')^2 ds' dt'$$

which is finite for all $a > 0$. This implies that h vanishes on $[0, T] \times [0, T]$. \square

Proof. (i) X satisfies the condition (C), and by Theorem (2.1), it admits a canonical Volterra representation with a standard Brownian motion W and a kernel of the form of $k(t, s) = t^{\beta-\frac{1}{2}} F\left(\frac{s}{t}\right)$, $F \in L^2(\mathbb{R}_+, du)$ vanishing on $(1, \infty]$. By using Fubini theorem, (3.3) gives

$$\tilde{X}_t = X_t - \int_0^t \int_s^t k(t, u) l(u, s) du dW_s, \quad 0 \leq t \leq T,$$

which proves the claim.

ii) Suppose that \tilde{X} is β -self-similar. From (i), \tilde{X} has the representation

$$\tilde{X}_t = \int_0^t \left(k(t, s) - t^{\beta-\frac{1}{2}} z(t, s) \right) dW_s, \quad 0 \leq t \leq T,$$

which is a canonical Volterra representation. Indeed, if \mathcal{L} denotes the Volterra integral operator associated with the Volterra kernel $l(t, s)$, the integral operator $\mathcal{K} - \mathcal{K}\mathcal{L} = \mathcal{K}(\mathcal{I} - \mathcal{L})$ that corresponds to the Volterra kernel $k(t, s) - t^{\beta-\frac{1}{2}} z(t, s)$ is also a Volterra integral operator, [7]. Here, \mathcal{I} denotes the Identity operator. In particular, if we let $f \in L^2([0, T])$ be such that $\mathcal{K}(\mathcal{I} - \mathcal{L})f = 0$. By (i) in Remark 1.1, the operator \mathcal{K} is injective, hence, $(\mathcal{I} - \mathcal{L})f = 0$, i.e., $\mathcal{L}f = f$. Therefore, the Volterra integral operator \mathcal{L} admits an eigenvalue, which is a contradiction by (ii) in Remark 1.1. So, $f \equiv 0$.

Now, using the fact that $H_{\tilde{X}}(t) = H_X(t)$ for all t , \tilde{X} satisfies also the condition (C), and by Theorem (2.1), the canonical kernel $k(t, s) - t^{\beta-\frac{1}{2}} z(t, s)$ is $(\beta - \frac{1}{2})$ -homogeneous. For $a > 0$, we write

$$k(t, s) - t^{\beta-\frac{1}{2}} z(t, s) = a^{\beta-\frac{1}{2}} \left(k\left(\frac{t}{a}, \frac{s}{a}\right) - t^{\beta-\frac{1}{2}} z\left(\frac{t}{a}, \frac{s}{a}\right) \right),$$

which implies that $z(t, s) = z(\frac{t}{a}, \frac{s}{a})$, and by the change of variable, we have

$$\int_s^t F\left(\frac{u}{t}\right) l(u, s) du = \int_{\frac{s}{a}}^{\frac{t}{a}} F\left(\frac{u}{\frac{t}{a}}\right) l\left(u, \frac{s}{a}\right) du = \int_s^t F\left(\frac{v}{t}\right) \frac{1}{a} l\left(\frac{v}{a}, \frac{s}{a}\right) dv, \quad s < t,$$

which equivalent to

$$\int_0^t F\left(\frac{u}{t}\right) l(u, s) du = \int_0^t F\left(\frac{u}{t}\right) \frac{1}{a} l\left(\frac{u}{a}, \frac{s}{a}\right) dv, \quad s < u < t.$$

Taking derivatives with respect to t on both sides, and since $F\left(\frac{u}{t}\right) \neq 0$, we obtain

$$l(u, s) = \frac{1}{a} l\left(\frac{u}{a}, \frac{s}{a}\right), \quad s < u,$$

which means that l is homogeneous with degree (-1) . By applying the Lemma 3.2, we get $l \equiv 0$.

If $l \equiv 0$, we have $\mathbb{E}(\tilde{X}_t \tilde{X}_s) = \mathbb{E}(X_t X_s)$ which means that $\tilde{X} \stackrel{d}{=} X$. Therefore, \tilde{X} is β -self-similar. \square

Remark 3.3. The importance of the condition (C) in Proposition (3.1) can be seen in the case of the fBm with index $H = 1$, i.e. $B_t^H = tB_1^H$, $0 \leq t \leq T$. Here the condition (C) fails. Since fBm is Gaussian, each process is determined by its covariance $\mathbb{E}(B_t^H B_s^H) = ts \mathbb{E}((B_1^H)^2)$. However, the laws of processes that correspond to different values of $\mathbb{E}((B_1^H)^2)$ are equivalent, on the other hand, these laws are different.

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